Let $Q(f) := f(0) + f'(0)x + \frac{f''(0)}{2}x^2$.

Show $Q(fg) = Q\left(Q(f)Q(g)\right)$.

Cheat sheet:

$(fg)' = f'g + fg'$

$(fg)'' = f''g + 2f'g' + fg''$

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$$Q(f) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2$$

$$\Rightarrow Q(fg) = fg(0) + (fg)'(0)x + \frac{(fg)''(0)}{2}x^2$$

$$\Rightarrow Q(fg) = fg(0) + [f'g(0) + fg'(0)]x$$
$$+ \left[\frac{f''g(0) + 2f'g'(0) + fg''(0)}{2}\right]x^2$$

$Q(f)Q(g)$

$$= \left(f(0)+f'(0)x+\frac{f''(0)}{2}x^2\right)\left(g(0)+g'(0)x+\frac{g''(0)}{2}x^2\right)$$

$$= f(0)g(0) + f'(0)g(0)x + \frac{f''(0)g(0)}{2}x^2$$
$$+ f(0)g'(0)x + f'(0)g'(0)x^2 + \frac{f''(0)g'(0)}{2}x^3$$
$$+ \frac{f(0)g''(0)}{2}x^2 + \frac{f'(0)g''(0)}{2}x^3 + \frac{f''(0)g''(0)}{4}x^4$$

$$\Rightarrow Q(Q(f)Q(g))$$
$$= f(0)g(0) + (f'(0)g(0)+f(0)g'(0))x$$
$$+ \left(\frac{f''(0)g(0)}{2} + \frac{f(0)g''(0)}{2} + f'(0)g'(0)\right)x^2$$

Higher order terms more than 2 are dropped

$\therefore Q(fg) = Q(Q(f)Q(g))$. $\square$